

High-precision early mission narrow-angle science with the Space Interferometry Mission

Stuart Shaklan, Mark H. Milman, Xiaopei Pan

Jet Propulsion Laboratory, California Institute of Technology
4800 Oak Grove Dr., Pasadena, CA 91109

ABSTRACT

We have developed a technique that allows SIM to measure relative stellar positions with an accuracy of 1 micro-arcsecond at any time during its 5-yr mission. Unlike SIM's standard narrow-angle approach, Gridless Narrow Angle Astrometry (GNAA) does not rely on the global reference frame of grid stars that reaches full accuracy after 5 years. GNAA is simply the application of traditional single-telescope narrow angle techniques to SIM's narrow angle optical path delay measurements. In GNAA, a set of reference stars and a target star are observed at several baseline orientations. A linearized model uses delay measurements to solve for star positions and baseline orientations. A conformal transformation maps observations at different epochs to a common reference frame. The technique works on short period signals (P =days to months), allowing it to be applied to many of the known extra-solar planets, intriguing radio/X-ray binaries, and other periodic sources. The technique's accuracy is limited in the long-term by false acceleration due to a combination of reference star and target star proper motion. The science capability 1 micro-arcsecond astrometric precision – is unique to SIM.

Keywords: Space Interferometry Mission, relative astrometry, extra-solar planets, binary stars.

1. INTRODUCTION

The Space Interferometry Mission (SIM) is capable of detecting and measuring the mass of terrestrial planets around stars other than our own. It can measure the mass of black holes and the visual orbits of radio and x-ray binary sources. There is no doubt that with the successful conclusion of the SIM mission, the Universe will contain many newly discovered planets and many breakthroughs in our understanding of complex astrophysical processes.

SIM achieves its high precision in the so-called "narrow-angle" regime. This is defined by a small (1°) diameter field in which the position of a target star is measured with respect to a set of reference stars. The observation is performed in two parts: first, SIM observes a grid of stars that spans the full sky. After a few years, repeated observations of the grid allow one to determine the orientation of the interferometer baseline. Second, throughout the mission, SIM periodically observes in the narrow-angle mode. Every narrow-angle observation is linked to the grid to determine the precise attitude and length of the baseline.

The narrow angle process demands patience. It is not until five years after launch that the SIM data can be reduced to yield the ultimate accuracy. The accuracy is best at mid-mission, and a factor of 2 worse at the beginning and end of the mission.¹

This manuscript describes a process for obtaining high-precision narrow-angle measurement with SIM at any epoch. We show through analysis and simulations that SIM can obtain uas accuracy and can detect extra-solar planets with a few days of observation, and without relying on a high-precision grid of reference stars. The process can be applied as early as during the first six months of in-orbit calibration (IOC). We call this technique Gridless Narrow Angle Astrometry, or GNAA.

The motivations are two-fold. First, and obviously, it is an insurance policy against a catastrophic mid-mission failure. Second, early results and a technique that can duplicate those results throughout the mission will give the mission analysts important experience in the proper use and calibration of SIM.

2. GRIDLESS NARROW ANGLE ASTROMETRY (GNAA)

2.1. SIM's Baseline Approach

The process of performing narrow angle astrometry with SIM is in principle straightforward: SIM derives the baseline orientation, length, and delay constant by observing stars whose positions are known to a few microarcsec (uas). These are referred to as "grid stars" as they are part of the high-precision all-sky grid that is the basis for SIM's narrow-angle and wide-angle astrometry (reference: Swartz). It then observes the target and nearby reference stars, ultimately computing the position of the target relative to the reference stars along the direction of the baseline. The baseline is then rotated (nominally within a few days) and the stars are re-observed to measure the position on the orthogonal axis.

Narrow angle accuracy is limited by the number of grid stars and the accuracy of their measured positions. Grid stars are spread over a 15-degree diameter "tile" while the target and reference are confined to a 1-degree field near the center of the tile. To first order, for a given grid-star angular error e , the narrow angle error is $\approx e/15$, but this is subject to the intermediate step of baseline determination. Simulations show that on average 6 grid stars are required per tile if grid star errors of 4 uas r.m.s. are to contribute < 0.3 uas r.m.s. to the narrow angle solution.

The narrow angle measurement process does not achieve its full accuracy until the end of the mission because grid star positions, proper motions, and parallaxes are not known to their potential accuracy until then. A first grid star campaign is planned during the initial 6-month in-orbit checkout (IOC), but it will be ≈ 1 year before an accurate parallax (≈ 15 uas) is determined over a significant fraction of the sky. Thus the SIM science teams do not plan to report accurate narrow angle results in a timely fashion, even for known short-period systems. The baseline narrow angle technique relies on the grid star campaign, evolving in accuracy over the 5-year mission life.

2.2. Efficiency

Narrow angle measurements are scheduled in an efficient manner to take advantage of grid star observations and other objects of interest within a tile. The narrow angle measurements are chopped into 1-minute long observations (30 seconds of integration, 30 seconds of slew and acquisition) with 10 observations per star per hour. Observation of a target and 2 reference stars would then require 30 minutes per hour. Observation of 6 grid stars before and after the narrow angle measurements would use another 12 minutes. In theory a full 2-d measurement could be completed in about 1.5 hr.

Compared to the technique proposed below, the SIM narrow angle paradigm is more efficient. The new approach requires narrow angle observation of 4 reference and 1 target star, requiring the full hour (ten 1-minute long observations of 6 objects) to complete a cycle. Further, the new technique requires a minimum of 3 baseline pointings to make one 2-d measurement. After 3 hours, a narrow angle measurement is made on one interesting target. Thus the efficiency is reduced by a factor of 2, depending on how the grid stars are bookkept. The advantage of course is that the measurement is good to 1 uas after just 3 hours whereas the standard SIM approach requires >2 yr to make this claim.

2.3. A New Technique: GNAA

Narrow angle astrometry is nothing more than relative-angle astrometry; one measures the position of a target star with respect to a set of reference stars. In traditional narrow angle astrometry, a photographic plate or charge-coupled detector (CCD) stares at a field in successive observations. The field's reference stars are used to anchor a least-squares conformal transformation that matches the scale, rotation, translation, and potentially higher order angular terms into a common reference frame.² The transformation is then applied to the target star whose motion is observed relative to the reference frame.

The conformal transformation absorbs several instrumental parameters that are of no consequence to relative observational accuracy. Changes in the telescope focal length are seen as plate-scale changes while telescope pointing errors are simply field translations, and detector rotation results in field rotation. Because one only measures the relative positions of the stars, absolute scale of measured parameters is lost, but the relative accuracy is still as good as the a priori knowledge of the field, i.e. 100 milli-arcsec (mas) over 1 degree, or $3e-5$ if the reference star catalog positions are good to 100 mas. The minute absolute scale error when applied to the small reflex motion of the target stars is dwarfed by other factors (e.g. shot and detector noise).

GNAA with SIM is little more than the application of traditional narrow angle techniques to SIM's narrow angle optical path delay measurements. The technique, described in the following paragraphs, allows one to perform micro-arcsecond astrometry without solving for precise values of baseline length, baseline orientation, or the metrology constant term. In GNAA, a set of reference stars and a target star are observed at several baseline orientations. A linearized model is used to solve for star positions and baseline orientations. A conformal transformation is applied to relate the reference and target stars to a common reference frame.

As with narrow angle astrometry at a telescope, the conformal transformation absorbs SIM instrumental parameters. To first order baseline length errors cause field dilation, baseline orientation about the line-of-sight cause field rotation, baseline orientation errors orthogonal to the line-of-sight cause field translation and quadratic field distortion (focus and astigmatism), and the metrology constant term is a translation along the direction of the baseline. The conformal transformation solves the scale, rotation, and translation of the observed reference frame without requiring the intermediate step of exact baseline determination. The absolute scale is lost, but it is estimated with a precision approximately given by the *a priori* scale knowledge of the field size, as described above.

The most significant advantage of GNAA is that high-accuracy narrow-angle measurements can be made early in the mission. As there is no reliance on highly accurate grid star positions, the technique can be applied as soon as the instrument is calibrated. Thus SIM can obtain important scientific results soon after launch and throughout the early parts of the mission. Further, it is an ideal approach not only for early-mission science, but also for quick study and follow-up of compelling objects demanding immediate results. It is certain that some science will demand accurate results long before the 5-yr mission is concluded.

The disadvantages of GNAA are that it is observationally inefficient, and it suffers from false acceleration determination for long observations (a few uas over 5 years). Unlike the grid-based narrow angle technique that allows sharing of a tile between narrow-angle and other targets, GNAA requires 3 dedicated hours of observation to produce a uas 2-d measurement of the target star. At least 4 reference stars and 3 baseline orientations are needed. This technique is clearly not the choice for producing the bulk of narrow-angle science observations once normal operations begin. Further, GNAA is not the choice for observing periodic effects with periods 5 yr. The combination of arcsec/yr target star motion with milli-arcsec/yr reference star motion results in unobservable frame rotation that causes uas target star accelerations after several years.

3. ANALYSIS

The analysis is set up in two parts. First, we show that given a set of stars, baseline orientations and the corresponding delay measurements, there exists a transformation that allows one to determine the star positions from the delay measurements under certain conditions. We describe the null space of the solution, which consists of translation, rotation, scale, and quadratic displacement degrees of freedom. The relative star positions are solved modulo the null space.

Second, we show how observations at different epochs are mapped to a common epoch through a matrix that incorporates the aforementioned null space. Our error analysis reveals that the target star motion suffers from a long-term false acceleration caused by the proper motions of the target and reference stars. This was originally predicted by Shao,³ and our analysis confirms the predicted magnitude of the effect.

3.1. Proof of GNAA Principal

Let $D \subset \mathbf{R}^2$ be a subset of the open unit disc including the origin and let $\Psi : D \rightarrow S$, where S is the unit sphere:

$$\Psi(u, v) = (u, v, \gamma(u, v)); \quad \gamma(u, v) \equiv \sqrt{1 - u^2 - v^2} \quad (1)$$

Star positions in the field of regard are parameterized by Ψ , that is we write

$$s_i = (u_i, v_i, \gamma(u_i, v_i)). \quad (2)$$

Stellar positions are estimated using linearized equations about nominal *a priori* positions of the stars and baseline vectors. Within the linear approximation, the true and *a priori* star positions are related as

$$s_i = s_i^0 + \delta s_i, \quad (3)$$

where δs_i is a tangent vector to the sphere at s_i^0 .

δs_i is parameterized in the following way. Let $\Psi_*(u, v)$ denote the differential of the map Ψ at the point (u, v) . Then Ψ_* maps a tangent vector in D to a tangent vector on the sphere at $\Psi(u, v) = (u, v, \gamma(u, v))$ via

$$\Psi_*(u, v) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -u/\gamma & -v/\gamma \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \quad (4)$$

for any tangent vector $h = (h_1, h_2)$ in D .

In the narrow field, the set of delay measurements d_{ij} are generated via

$$d_{ij} = \langle s_i, b_j \rangle + c_j + \eta_{ji}; \quad i = 1, \dots, N, \quad j = 1, \dots, M \quad (5)$$

where b_j is the interferometer baseline vector, c_j is the interferometer constant term associated with the baseline vector, and η_{ji} is the noise in the measurement. Neither b_j nor c_j are known with sufficient precision, and hence they must either be estimated or eliminated from the set of equations to determine the science position vector.

Writing $b_j = b_j^0 + \delta b_j$, where b_j^0 is the *a priori* estimate of the baseline vector, (5) becomes (ignoring the noise term in the sequel)

$$d_{ij} = \langle s_i^0 + \delta s_i, b_j^0 + \delta b_j \rangle + c_j \quad (6)$$

$$= \langle s_i^0, b_j^0 \rangle + \langle \delta s_i, b_j^0 \rangle + \langle s_i^0, \delta b_j \rangle + c_j. \quad (7)$$

Hence,

$$d_{ij} - \langle s_i^0, b_j^0 \rangle = \langle \delta s_i, b_j^0 \rangle + \langle \delta b_j, s_i^0 \rangle + c_j, \quad (8)$$

where the left side of the equation is known: d_{ij} are the measurements and $\langle s_i^0, b_j^0 \rangle$ is the expected delay derived from *a priori* knowledge.

Now let Π denote the projection $\Pi(x_1, x_2, x_3) = (x_1, x_2)$ defined for any 3-vector $x = (x_1, x_2, x_3)$. (Thus when Π is restricted to the unit sphere, $\Pi = \Psi^{-1}$.) Using the tangent vector description we have

$$\langle \delta s_i, b_j^0 \rangle = \langle b_j^0, \Psi_*(u_i, v_i) h_i \rangle \quad (9)$$

$$= \langle \Psi_*^T(u_i, v_i) b_j^0, h_i \rangle \quad (10)$$

$$= \langle \Pi b_j^0, h_i \rangle + O(\max\{|u_i|, |v_i|\} |b_j^z| |h_i|), \quad (11)$$

where b_j^z denotes the third component of the vector b_j^0 . To get a handle on the magnitude of the O term, note that the narrow angle scenario restricts the field to a 1deg diameter so that $|u|, |v| < .01$. Then if the baseline is controlled to within one arcmin of the $u-v$ plane and the *a priori* error on the star position is on the order of 30mas, the O term contributes 2.5pm of delay error. Hence, there are no operational restrictions to deleting this term and assuming the model has the form

$$d_{ij} - \langle s_i^0, b_j^0 \rangle = \langle \Pi b_j^0, h_i \rangle + \langle \delta b_j, s_i^0 \rangle + c_j. \quad (12)$$

Next define the 2×2 rotation matrix E ,

$$E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad (13)$$

and consider the decomposition of δb_j as

$$\delta b_j = \epsilon_r \Pi b_j^0 + \epsilon_T E(\Pi b_j^0) + \delta b_j^z \quad (14)$$

where again δb_j^z denotes the third component of the vector δb_j . Thus we may write

$$\langle s_i^0, \delta b_j \rangle + c_j = \epsilon_r \langle \Pi b_j^0, \Pi s_i^0 \rangle + \epsilon_T \langle \Pi b_j^0, E^T(\Pi s_i^0) \rangle + c_j + \delta b_j^z \gamma_i. \quad (15)$$

Let $\rho_i^2 = u_i^2 + v_i^2$. Then $\gamma_i \approx 1 - \rho_i^2/2$, and

$$c_j + \delta b_j^z \gamma_i = c_j + \delta b_j^z - \delta b_z \rho_i^2 / 2. \quad (16)$$

Introducing the new constant $c'_j \equiv c_j + \delta b_j^z$, the delay equation becomes

$$d_{ij} - \langle s_i^0, b_j^0 \rangle = \langle \Pi b_j^0, h_i \rangle + \varepsilon_r \langle \Pi b_j^0, \Pi s_i^0 \rangle + \varepsilon_T \langle \Pi b_j^0, E^T(\Pi s_i^0) \rangle - \delta b_j^z \rho_i^2 / 2 + c'_j. \quad (17)$$

The unknown variables in the equation above include h_i , $i = 1, \dots, N$, ε_r , ε_T , δb_j^z , and c'_j . The external metrology subsystem tracks the change in the length of the science baseline vector without interruption during the generation of the entire set of delays in (17). Coupling this with the operational scenario that maintains the interferometer baseline vector in the $u-v$ plane, we may assume that ε_r represents a single unknown. This is to be contrasted with ε_T , δb_j^z , and c'_j that we allow to change for each baseline orientation. Hence each of these terms represents an M vector. Now let X denote the entire vector of unknowns:

$$X = \begin{bmatrix} h_1 \\ \vdots \\ h_N \\ \varepsilon_r \\ \varepsilon_T \\ \delta b^z \\ c \end{bmatrix}. \quad (18)$$

Then X is computed to be an $2N + 3M + 1$ vector. However, we will show below that there are a total of 6 redundant degrees of freedom. To characterize these redundant degrees of freedom we write the system of equations (17) as

$$d = A_d X, \quad (19)$$

and search for the null space of A_d (written $\text{null}(A_d)$). We will get these vectors just by eyeballing the situation.

By setting $\varepsilon_r = 0$, $\varepsilon_T = 0$, $\delta b_z = 0$, $h_i = (\kappa_1, \kappa_2)$, and $c'_j = -\langle \Pi b_j^0, h_i \rangle$, we see that this vector is in the null space for any choice of κ_i . For two independent choices of κ_1 and κ_2 , let u_1 and u_2 denote the corresponding vectors in $\text{null}(A_d)$. A third vector in the null space is obtained by setting $\varepsilon_T = 0$, $\delta b^z = 0$, $c'_j = 0$, $h_i = \Pi s_i^0$, and $\varepsilon_r = -1$. Denote this vector as u_3 . A fourth null space vector is constructed by taking $\varepsilon_r = 0$, $\delta b^z = 0$, $c'_j = 0$, $h_i = E^T(\Pi s_i^0)$, and $\varepsilon_T = -(1, \dots, 1)$. Let u_4 denote this vector. Two other null space vectors are obtained in the following way. Fix i in (17) and set ε_r , ε_T , $c' = 0$. For each i , consider solutions to the system of equations

$$\Pi b_j^0 h_i + \delta b_j^z \rho_i^2 / 2 = 0, \quad j = 1, \dots, M. \quad (20)$$

Let B denote the 2×2 matrix

$$B = \begin{bmatrix} \Pi b_1^0 \\ \Pi b_2^0 \end{bmatrix}, \quad (21)$$

where Πb_j^0 is row j . B is invertible so there exist δb_1^z and δb_2^z such that

$$B h_i + \rho_i^2 / 2 \begin{bmatrix} \delta b_1^z \\ \delta b_2^z \end{bmatrix} = 0, \quad (22)$$

with $h_i = \rho_i^2 / 2 (1, 0)$. By setting

$$\delta b_j^z = -\langle b_j^0, B^{-1} \begin{bmatrix} \delta b_1^z \\ \delta b_2^z \end{bmatrix} \rangle, \quad j = 3, \dots, M \quad (23)$$

we see that $h_i = (1, 0)$ and δb_j^z as defined in (22)–(23) satisfy (17) for all i . Thus the vector with these components is in $\text{null}(A_d)$. The same construction using $h_i = \rho_i^2 (0, 1)$ yields another vector in the null space. Designate these two vectors u_5 and u_6 .

Now introduce $\Gamma : \mathbf{R}^{2N+3M+1} \rightarrow \mathbf{R}^{NM}$ by

$$\Gamma X = \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix}, \quad (24)$$

and define $h^i = \Gamma u_i$, $i = 1, \dots, 6$. The h^i are simply the star position components of the null vectors. h^1 and h^2 correspond to a translation of the field, h^3 corresponds to a dilation of the field, h^4 is a rotation of the field, and h^5 and h^6 are quadratics in the u and v directions, respectively. So in general any two solutions of (19) with h and h' representing the star position components of the solutions are related as

$$h = h' + \sum_{i=1}^6 \alpha_i h^i, \quad (25)$$

for some scalars α_i . Let A^\dagger denote the pseudoinverse of A_d . Then the solution of the star position coordinates given by

$$\hat{h} = \Gamma A_d^\dagger d \quad (26)$$

is of fundamental interest. This solution has no component in the span of $\{h^i\}_{i=1,\dots,6}$ and is useful for characterizing the stability properties of the GNAA method. The error multipliers on the star positions, once the null space has been removed, are quite favorable (near unity) for reasonable geometries. In many applications of GNAA it is also possible to ignore the contribution of h^5 and h^6 . Conditions where this is possible include the cases where the field of regard is very small (much less than 1 degree), or when the reference stars are located on a thin annular disc about the center of the field. In this latter case the u_5 and u_6 are nearly indistinguishable from u_1 and u_2 .

The minimum number of stars and baseline orientations required to determine the star positions (modulo the null space) is fixed by the number of measurements, $N * M$, and the number of degrees of freedom (DOFs). We have fixed the baseline length for the M observations, so there are $2M + 1$ baseline DOFs, plus M metrology constant terms, to go along with the $2N - 6$ (6 being the dimension of the null space) stellar DOFs. Thus the number of stars and baselines is set by

$$NM \geq 2N + 3M - 5. \quad (27)$$

This is solved using a minimum of $N = 4$ stars and $M = 3$ baselines.

3.2. Astrometric Formulation

Over a small field of regard (1°) the curvature of the sphere can be effectively ignored, and it suffices to model points on the sphere as lying in a plane. Given a set of N reference stars, s_i , we will assume their motion is restricted to such a plane and obeys a law of the form

$$s_i(t) = s_i(0) + v_i t, \quad (28)$$

where t is time and v_i is the proper motion vector. We will actually assume that v_i represents an unknown proper motion, i.e. the residual between the true and *a priori* proper motion vectors. The target star, s_T , has a similar evolution:

$$s_T(t) = s_T(0) + v_T t + \phi(t). \quad (29)$$

For the target star we will assume that v_T is known; the objective is to identify $\phi(t)$.

GNAA works in the following way. The standard model for the positions of the reference and target stars from which estimates are derived from delay measurements has the form

$$\hat{s}_i(t) = \hat{s}_i(0) + h_i(t), \quad i = 1, \dots, N \quad \hat{s}_T(t) = \hat{s}_T(0) + t v_T + h_T(t), \quad (30)$$

where $\hat{s}_i(0), \hat{s}_T(0)$ are the estimates of the positions of the reference and target object at an initial time $t = 0$ *determined from the first set of measurements*, and $h_i(t)$ are correction terms determined from measurements. (So $h_i(0) = 0$.) The h_i can only be determined up to a translation, rotation, or scale factor. The latter ambiguity is due to the unknown absolute scale of the instrument. The GNAA principal (proved in the previous section) implies that there exists a transformation $A(t)$,

$$A(t)s = \kappa(t) + \epsilon(t)s + w(t)Es, \quad (31)$$

such that

$$s_i(t) - \hat{s}_i(t) = A(t)\hat{s}_i(t) \quad (32)$$

for all $i = 1, \dots, N, T$ and for any set of estimates of the star positions taken at any time $t \geq 0$. Here κ is a two-vector, ε and w are scalars, and E is the matrix

$$E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (33)$$

Now (32) also holds for $t = 0$, i.e.

$$s_i(0) - \hat{s}_i(0) = A(0)\hat{s}_i(0). \quad (34)$$

It is easy to see that the set of maps of the form (31) are closed under addition, so in particular $A_t = A(t) - A(0)$ is also of the form (31). With this in mind we compute using (32) and (34)

$$s_i(t) - s_i(0) = \hat{s}_i(t) - \hat{s}_i(0) + A(t)\hat{s}_i(t) - A(0)\hat{s}_i(0) \quad (35)$$

$$= \hat{s}_i(t) - \hat{s}_i(0) + A_t\hat{s}_i(0) \quad (36)$$

after using $\hat{s}_i(t) = \hat{s}_i(0) + h_i(t)$ and deleting terms of second order. Because the proper motion of the target star is very large, we will retain its second order contribution below to obtain

$$s_T(t) - s_T(0) = \hat{s}_T(t) - \hat{s}_T(0) + A_t[\hat{s}_T(0) + tv_T] + tA(0)v_T. \quad (37)$$

The objective is to determine the left side of (37) above; this is the motion of the target star. On the right side above, the unknowns are the transformations A_t and $A(0)$. $A(0)$ only contributes a linear term, while A_t can be determined, in principle, from (36). However, an error will arise because A_t cannot be determined exactly from (36) since the model of the left side of the equation contains proper motion errors.

3.3. Error Analysis.

Introduce the $2N \times 4$ matrix X

$$X = \begin{bmatrix} I_{2 \times 2} & s_1 & Es_1 \\ \vdots & \vdots & \vdots \\ I_{2 \times 2} & s_N & Es_N \end{bmatrix}, \quad (38)$$

and the 4 component parameter vector p that characterizes A_t via

$$A_t s = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + p_3 s + p_4 E s. \quad (39)$$

Then (36) can be written as

$$tv = \hat{s} + Xp, \quad (40)$$

where

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad \text{and} \quad \hat{s} = \begin{bmatrix} \hat{s}_1(t) - \hat{s}_1(0) \\ \vdots \\ \hat{s}_N(t) - \hat{s}_N(0) \end{bmatrix}. \quad (41)$$

Thus, $p = X^\dagger [tv - \hat{s}]$, where X^\dagger denotes the pseudoinverse of X . But since there is no knowledge of the proper motions v_i , the estimated transformation uses $\hat{p} = -X^\dagger \hat{s}$ instead, and an error of $t\delta p$ arises where

$$\delta p = X^\dagger v. \quad (42)$$

Let \hat{A}_t denote the transformation in (39) using \hat{p} , and again consider (37). The last two terms on the right are the corrections to the position estimate differences using the transformations A_t and $A(0)$. However, the best we can do is use \hat{A}_t . The resulting error in the (change in) position of the target star, call it $\delta s_T(t)$ is,

$$\delta s_T(t) = (A_t - \hat{A}_t)[\hat{s}_T(0) + tv_T] + tA(0)v_T. \quad (43)$$

From (39), (42), and (43) we see that

$$\delta s_T(t) = t \begin{bmatrix} \delta p_1 \\ \delta p_2 \end{bmatrix} + t \delta p_3 [\hat{s}_T(0) + t v_T] + t \delta p_4 E [\hat{s}_T(0) + t v_T] + t A(0) v_T. \quad (44)$$

Now, from (42) it is evident that δp is linear with time; hence the acceleration error α is given by the coefficients of the quadratic (in t) terms above:

$$\alpha = 2[\delta p_3 v_T + \delta p_4 E v_T]. \quad (45)$$

Since v_T and $E v_T$ are orthogonal with $|E v_T| = |v_T|$,

$$|\alpha| = 2|v_T| |\delta q|, \quad \text{where} \quad \delta q = \begin{bmatrix} \delta p_3 \\ \delta p_4 \end{bmatrix}. \quad (46)$$

Let T denote the period of observation. After performing a linear least squares fit to the displacement due to acceleration, $t \rightarrow \alpha t^2/2$ over the period $0 \leq t \leq T$, the residual error, $r(t)$ is the quadratic

$$r(t) = |v_T| |\delta q| \left\{ -\frac{1}{6} T^2 + Tt - t^2 \right\} \quad (47)$$

with a peak-to-peak displacement, δ ,

$$\delta = |v_T| |\delta q| T^2/4. \quad (48)$$

It remains to characterize δq . Without loss of generality we may assume the origin of our coordinate system is the barycenter of the reference star positions, i.e.

$$\sum_{i=1}^N \hat{s}_i(0) = 0. \quad (49)$$

By linearity we then also have

$$\sum_{i=1}^N E \hat{s}_i(0) = 0. \quad (50)$$

Since X has full rank, $X^\dagger = (X^T X)^{-1} X^T$. Now define the 2×2 matrices, S_i as

$$S_i = \begin{bmatrix} s_i & E s_i \end{bmatrix}. \quad (51)$$

Note that $X^T X$ is the 4×4 matrix

$$X^T X = \begin{bmatrix} N I_{2 \times 2} & \sum_{i=1}^N S_i \\ \sum_{i=1}^N S_i^T & \sum_{i=1}^N S_i^T S_i \end{bmatrix}. \quad (52)$$

But $\sum S_i = 0$ by (49) and (50), and

$$\sum_{i=1}^N S_i^T S_i = \sum_{i=1}^N |s_i|^2 I_{2 \times 2}. \quad (53)$$

Hence, $X^T X$ is the diagonal matrix

$$X^T X = \begin{bmatrix} N I_{2 \times 2} & 0 \\ 0 & \sum_{i=1}^N |s_i|^2 I_{2 \times 2} \end{bmatrix}. \quad (54)$$

Thus,

$$\delta q = \frac{1}{\sum_{i=1}^N |s_i|^2} \left[\frac{\sum_{i=1}^N s_i^T v_i}{\sum_{i=1}^N |(E s_i)^T v_i|} \right], \quad (55)$$

and

$$|\delta q| = \frac{1}{\sum_{i=1}^N |s_i|^2} \left\{ \left[\sum_{i=1}^N s_i^T v_i \right]^2 + \left[\sum_{i=1}^N (E s_i)^T v_i \right]^2 \right\}^{1/2}. \quad (56)$$

Expressions (47,48), (55) and (56) expose the relevant parameters of the peak-to-peak quadratic error. For example, if we assume that the proper motions $\{v_i\}$ are zero mean independent random vectors with common variance σ_v^2 , we find the peak-to-peak displacement is

$$\delta_{\text{random}} = \frac{\sqrt{2}|v_T|\sigma_v T^2}{4\sqrt{\sum_{i=1}^N |s_i|^2}}. \quad (57)$$

Hence, a $1/\sqrt{N}$ improvement in the error is realized with increasing the number of reference stars, while a linear improvement is obtained by increasing the separation of the reference stars. In general, without the statistical assumption, there is no advantage to using more stars.

3.4. Example of False Acceleration

Shao³ created the following example: the reference frame contains two stars, one at the origin, the other on the v-axis. They are moving with an apparent velocity of 1 milli-arcsec yr⁻¹ in opposite directions along the u-axis. Thus the reference frame has a non-zero angular momentum. The target star is located off to the side and moves with a velocity of 1 arcsec yr⁻¹ in the v direction. In our formulation, this is expressed by

$$s_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad s_2 = \frac{\pi}{180} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad s_T = \frac{\pi}{180} \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}, \quad (58)$$

and

$$v_1 = \frac{4.86 \times 10^{-9}}{\text{yr}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \frac{4.86 \times 10^{-9}}{\text{yr}} \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad v_T = \frac{4.86 \times 10^{-6}}{\text{yr}} \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \quad (59)$$

With respect to barycentric reference star coordinates, we have

$$s_1 = \frac{\pi}{180} \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}, \quad s_2 = \frac{\pi}{180} \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \quad (60)$$

Using these values in (48) and (56) together with a 5 year observation period, $T = 5$, leads to the displacement

$$\delta \approx 3.48 \mu\text{as}. \quad (61)$$

4. SIMULATION

We demonstrate the GNAA technique with the simulation of 55 Cnc (P=14.6 d) described here. The simulation shows that given a set of stars R with only a coarse a priori knowledge of their positions, the target star position can be measured with ≈ 1 mas accuracy for extended periods of time.

Based on the radial velocity measurements of the innermost planet around 55 Cnc,⁴ the system is expected to have an astrometric reflex amplitude of 8 mas for the inner planet. While we have not simulated the second component (P=44.3 d, a=0.24 AU⁵), it too is detectable by GNAA with an amplitude of ≈ 4.5 mas. The outermost planet, with its 14-yr period and > 2 mas signal, is easily detectable (though only a fraction of the orbit is observed during the SIM lifetime), but only because the signal is \gg the false acceleration due to frame rotation. The outer planet would be detected during the normal course of SIM narrow-angle observations – it is not a candidate for GNAA observations. However, the inner planet would NOT be detected. The nominal SIM narrow-angle observational scenario calls for ≈ 10 visits/target/yr, compared to 25 orbits/yr for the inner planet. The second planet might be detected (8 orbits/yr) depending on the details of SIM scheduling. Without GNAA measurements, additional planets with signals of a few mas would be masked by the ‘noise’ from the unseen inner planets.

The set of stars R was chosen over a 1° field-of-regard (FOR). The stars in R have an a priori positional knowledge of 100 mas. This could be easily obtained using ground based observations with a small telescope for stars not in the Hipparcos or Tycho catalogs. R is observed at 3 baseline orientations such that the baseline is rotated about the line of sight s in 60° steps. The 60° steps are assumed to be accurate to $\pm 1^\circ.5$ as is the positioning of the baseline cant angle (out of the $s \times B$ plane). The baseline orientation is assumed to be known to 1 arcsec with information provided by star

trackers, siderostat encoders, and coarse estimates of delay line position. Stars with $V < 10.5$ are assumed so that the delay measurement noise is consistent with the SIM narrow angle error budget.

A crucial assumption in this study is the regularization of B . Imperfect spacecraft attitude control allows the baseline to drift by up to ≈ 1 arcsec. Regularization describes the process of estimating the mean value of B using information from the guide interferometers and knowledge of the baseline geometry (guide baselines relative to the science baseline). The accuracy of this assumption is well understood by the SIM science team and is consistent with narrow-angle observational requirements. This assumption is always assumed in the standard narrow angle scenario as well.

At the microarcsecond level, the reference frame is deforming itself; the reference and target stars are moving linearly in random directions due to proper motion, and they are moving in functionally predictable trajectories due to parallax. The amplitude of the parallax is not known. Our simulation includes random (but linear in time) 10 mas/year motion per reference star, and 1 arcsec/year motion of the target star. Curvature (in particular for the target star) of the motion due to parallax is not included because the science program searches for known short period signals that are easily decoupled from the annual parallax.

SIM's metrology will occasionally break. In the conventional (grid-based) narrow angle approach, it is allowed to break with every ≈ 5 tiles; more frequent interruptions result in a poor absolute grid solution (the grid becomes too "elastic"). With GNAA, the internal metrology is allowed to break with every new baseline orientation (roughly every hour), resulting in a new c'_j term for each orientation in (17). But (17) also shows us that ϵ_r is independent of j – that is, the baseline length error is fixed for the M baseline orientations. Thus, the external metrology must not break until the 3 baseline orientations are complete, or ≈ 3 hr.

In summary the simulation includes the following effects:

Initial knowledge error of reference frame and target star positions	100 mas rms (each axis)
Baseline positioning accuracy	1°.5 rms
Knowledge of baseline pointing	1 arcsec rms (each axis)
Baseline absolute length knowledge	10 microns rms
Noise per delay measurement	50 pm rms
Constant term knowledge error	10 um rms
Proper motion of stars	10 mas/yr rms
Number of baseline orientations/2-D measurement:	3
Number of reference stars	5
External metrology break (baseline length measurement)	every 3 hrs
Constant term metrology break	every hour

The simulation shows that GNAA can measure changes in the position of the target relative to the reference frame at the micro-arcsecond level. Even for extended periods in which the reference frame deforms by 10 mas, the noise level remains ≈ 1 micro-arcsecond.

The astrometric precision depends upon the distribution of stars in the reference set, the accuracy of the a priori positions of the reference frame and baseline, and the noise level per measurement. However, simulations show that by far the most sensitive parameter is the delay noise per measurement (50 pm rms). When this is set to zero (no delay error), and all other parameters remain the same, the rms astrometric error is reduced to 0.23 uas. On the other hand, assuming perfect reference star knowledge and perfect baseline vector knowledge (but 50 pm rms noise) improves the precision from 1.0 to 0.95 uas. As the field size grows (in particular, the diameter of R'), sensitivity to baseline and reference star knowledge increases.

5. FUTURE WORK

The GNAA technique has been studied in some detail but much work remains. The GNAA process ultimately needs to be described in the same analytical framework as the SIM grid and standard narrow angle reductions. Noise propagation and parametric sensitivities will be studied. In doing this, one hopes to describe an optimized process that makes more efficient use of grid stars and precious narrow angle observing time.

This work was performed at the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration.

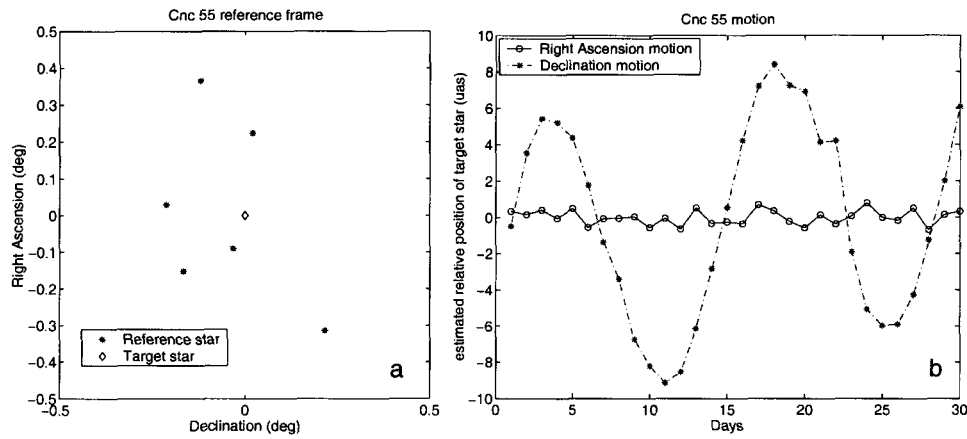


Figure 1. Fig. 1. a) is 55 Cnc and the surrounding field of reference stars. Only the nearest bright reference stars (shown) are used. b) shows the result of 30 daily observations of 55 CNC. Each day, the baseline is oriented to 3 positions rotated 60° about the line of sight to the target. For demonstration purposes, the 8 uas signal is assumed to be a pure North-South sine wave. The R.A. axis thus indicates the noise level. Given the simulation parameters described above and this particular set of reference stars, the astrometric precision is 1.0 uas on each axis for each 3-baseline (3-hour long) observation. The reference frame has deformed by up to 300 mas due to proper motions of the reference stars. The relative proper motion of the target star has been fitted and removed.

REFERENCES

1. Loiseau, S., Malbet, F., "Global Astrometry with OSI," *Astron. Astroph. Suppl.*, 116, 373-380 (1996).
2. Eichhorn, H., Williams, C., "On the Systematic Accuracy of Photographic Astrometric Data," *AJ* 68 221-231 (1963).
3. M. Shao, private communication, 2001.
4. Butler, R.P., et al "Three New 51 Peg-Like Planets," *Astrophys. J. Letters*, 474, L115 (1997).
5. Marcy, G., et al, submitted to *ApJ*. (2002).